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A Companion to Jensen–Steffensen's Inequality

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Jensen's inequality for convex functions can be stated as follows:

Suppose that f is convex on (a, b). Then for $x_1,..., x_n$ in (a, b) and $p_1,..., p_n \geqslant 0$, $P_n = \sum_{i=1}^n p_i > 0$,

$$f\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^n p_i f(x_i). \tag{1}$$

The following result is also known:

Suppose that f is convex on (a, b), $a < x_1 \leqslant \cdots \leqslant x_n < b$ and

$$0 \leqslant \sum_{i=1}^k p_i = P_k \leqslant P_n \ (1 \leqslant k \leqslant n-1), \qquad P_n > 0. \tag{2}$$

Then (1) again holds.

This is the well-known Jensen–Steffensen inequality.

Slater [1] proved the following companion to Jensen's inequality:

Suppose that f is convex and nondecreasing (nonincreasing) on (a, b). Then for $x_1,..., x_n \in (a, b)$, $p_1,..., p_n \geqslant 0$, $p_1 + \cdots + p_n > 0$, and $p_1 f'_+(x_1) + \cdots + p_n f'_+(x_n) \neq 0$, we have

$$\frac{1}{P_n}\sum_{i=1}^n p_i f(x_i) \leqslant f\left(\sum_{i=1}^n p_i x_i f'_+(x_i) \Big/ \sum_{i=1}^n p_i f'_+(x_i)\right). \tag{3}$$

An integral analog of this result was also given. Both results remain true if at any occurence of $f'_+(x)$ we write instead any value in the interval $[f'_-(x), f'_+(x)]$.

First, we note the following simple generalization of Slater's above result:

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THEOREM 1. *Suppose that f is convex on* (a, b). *If, for* $x_1, \ldots, x_n \in (a, b)$, $p_1, \ldots, p_n \geqslant 0$, $p_1 + \cdots + p_n > 0$, *we have*

$$\sum_{i=1}^{n} p_i f'_+(x_i) \neq 0, \quad \sum_{i=1}^{n} p_i x_i f'_+(x_i) \Big/ \sum_{i=1}^{n} p_i f'_+(x_i) \in (a, b), \tag{4}$$

then (3) *holds.*

The proof is similar to that in [1].

Now we give a companion to Jensen–Steffensen's inequality:

THEOREM 2. *Suppose that f is convex on* (a, b) *and* $a < x_1 \leqslant \cdots \leqslant x_n < b$. *If* (2) *and* (4) *hold, then so does* (3).

Proof. For arbitrary $x, y \in (a, b)$ we have

$$f(y) - f(x) \geqslant (y - x) f'_+(x), \tag{5}$$

i.e.,

$$f(x) - f(y) \leqslant (x - y) f'_+(x). \tag{6}$$

Therefore

$$\Delta_i = f(A) - f(x_i) - f'_+(x_i)(A - x_i) \geqslant 0 \qquad (i = 1, \ldots, n),$$

where

$$A = \sum_{i=1}^{n} p_i x_i f'_+(x_i) \Big/ \sum_{i=1}^{n} p_i f'_+(x_i).$$

Suppose that $A \in [x_k, x_{k+1}]$ $(k \in (1, \ldots, n-1))$. If $x_i \leqslant x_{i+1} \leqslant A$, then, using (5), we obtain

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\geqslant (x_{i+1} - x_i) f'_+(x_i) \\ &= (A - x_i) f'_+(x_i) - (A - x_{i+1}) f'_+(x_i) \\ &\geqslant (A - x_i) f'_+(x_i) - (A - x_{i+1}) f'_+(x_{i+1}), \end{aligned}$$

namely,

$$f(A) - f(x_i) - (A - x_i) f'_+(x_i) \geqslant f(A) - f(x_{i+1}) - (A - x_{i+1}) f'_+(x_{i+1}),$$

i.e.,

$$\Delta_i \geqslant \Delta_{i+1}.$$

Similarly, if $A \leqslant x_i \leqslant x_{i+1}$, then, from (6), we have

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\leqslant (x_{i+1} - x_i) f'_+(x_{i+1}) \\ &= (A - x_i) f'_+(x_{i+1}) - (A - x_{i+1}) f'_+(x_{i+1}) \\ &\leqslant (A - x_i) f'_+(x_i) - (A - x_{i+1}) f'_+(x_{i+1}), \end{aligned}$$

i.e.,

$$\Delta_i \leqslant \Delta_{i+1}.$$

Therefore we have

$$\begin{aligned} \sum_{i=1}^{n} p_i \Delta_i &= \sum_{i=1}^{k} p_i \Delta_i + \sum_{i=k+1}^{n} p_i \Delta_i \\ &= \Delta_k P_k + \sum_{i=1}^{k-1} P_i(\Delta_i - \Delta_{i+1}) \\ &\quad + \Delta_{k+1} \bar{P}_{k+1} + \sum_{i=k+2}^{n} \bar{P}_i(\Delta_i - \Delta_{i-1}) \\ &\geqslant 0 \quad (\bar{P}_k = P_n - P_{k-1}, k = 2, 3, ..., n; \bar{P}_1 = P_n), \end{aligned}$$

i.e.,

$$f(A) P_n - \sum_{i=1}^{n} p_i f(x_i) = \sum_{i=1}^{n} p_i \Delta_i \geqslant 0,$$

which is the inequality (3).

If $A \in (a, x_1)$, then Δ_i is nonnegative and nonincreasing for $i = 1, ..., n$. So we have

$$\sum_{i=1}^{n} p_i \Delta_i = \Delta_n P_n + \sum_{i=1}^{n-1} P_i(\Delta_i - \Delta_{i-1}) \geqslant 0,$$

i.e., (3) is again valid. Similarly we can prove (3) if $A \in (x_n, b)$.

Remarks. 1°. If

$$0 \leqslant \sum_{i=1}^{k} p_i f'_+(x_i) \leqslant \sum_{i=1}^{n} p_i f'_+(x_i) \qquad (1 \leqslant k \leqslant n-1),$$

then $A \in [x_1, x_n]$. For a nondecreasing function f and nonnegative p_i we have Slater's result.

2°. Using similar proofs, we can give integral analogs of Theorems 1 and 2.

Reference

1. M. L. Slater, A companion inequality to Jensen's inequality, *J. Approx. Theory* **32** (1981), 160–166.